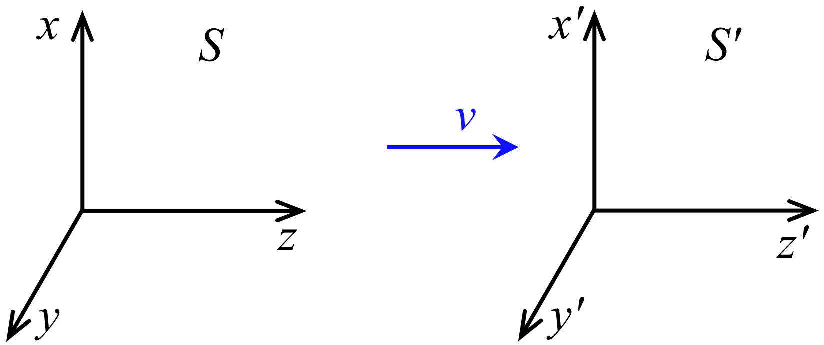
Relativistic kinematics: Lorentz Transformations



Speed v of S' w.r.t. the S along z-axes

$$\beta = v/c\ ,\ \ \gamma = 1/\sqrt{1-\beta^2}$$

Lorentz boost. γ – Lorentz factor

- Space and time coordinates have different values when measured in different inertial frames moving wrt one another. Relation is described by Lorentz Transformation (LT)

$$\begin{aligned} x' &= x \\ y' &= y \\ z' &= \gamma\,(z - \beta ct) \\ ct' &= \gamma\,(ct - \beta z) \end{aligned} \qquad \begin{aligned} x &= x' \\ y &= y' \\ z &= \gamma\,(z' + \beta ct) \\ ct &= \gamma\,(ct' + \beta z) \end{aligned}$$

At small speeds, $\beta = v/c \ll 1, \gamma = 1$, reduces to Galilean transformation

$$z' = \gamma\,(z - \beta ct) = z - vt$$
$$t' = t - v/c^2\, z = t$$

- Lengths perpendicular to the direction of motion are unaffected by LTs
- Length of an object along its direction of motion is related to its length at rest (S' frame) as $\Delta z = \Delta z'/\gamma$ ← length contraction
- Time interval of a moving clock ←→ elapsed time at rest: $\Delta t = \gamma \Delta t'$ ← time dilation
- Definition: “proper time”, τ, the lifetime of a particle in its rest frame
→ Particle’s lifetime in the frame in which it is moving is: $t = \gamma\tau$ and $t \geq \tau$

Relativistic kinematics: four-vectors, invariants, E, $\mathbf{p}$, $\mathbf{v}$

- Most general LT has simplest form in terms of 4-vectors: $a = (a_0, a_1, a_2, a_3) = (a_0, \mathbf{a})$
- Space-time and momentum 4-vectors: $x = (ct, \mathbf{x}) = (ct, \mathbf{r})$, $P = (E/c, \mathbf{p})$
- Scalar product of two 4-vectors defined as $ab \equiv a_0 b_0 - \mathbf{ab}$ is invariant under LT. In particular, LT preserves the quantities

$$a^2 = a^2_0 - \mathbf{a}^2,\ x^2 = (ct)^2 - x^2 - y^2 - z^2,\ P^2 = (E/c)^2 - p^2_x - p^2_y - p^2_z$$

LT implies

$$a'_1 = x$$
$$a'_2 = y$$
$$a'_3 = \gamma\,(z - \beta ct)$$
$$a'_0 = \gamma\,(ct - \beta z)$$

- Basic 4-vector in particle kinematics is the four-momentum, $P \equiv mu$, with $u = \gamma(c, \mathbf{v})$, where m is the rest mass, u is the 4-velocity and $\mathbf{v}$ is usual 3-velocity, and $v \equiv |\mathbf{v}|$
- Given two definitions for momentum 4-vectors: $P = mu = (m\gamma c, m\gamma\mathbf{v}) = (E/c, \mathbf{p})$

$$\Rightarrow\ E = \gamma mc^2,\ \ \mathbf{p} = \gamma m\mathbf{v},\ \ \mathbf{v} = \frac{c^2}{E}\mathbf{p}$$

cf. Newtonian mechanics: $E = \frac{1}{2}mv^2$, $\mathbf{p} = m\mathbf{v}$, $E = \frac{p^2}{2m}$

- In terms of the total energy E and the 3-momentum $\mathbf{p}$: $P = (E/c, \mathbf{p})$

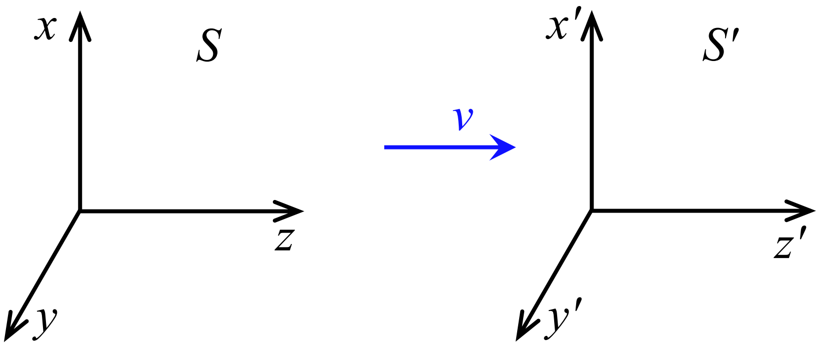
$$\Rightarrow P^2 = (E/c)^2 - \mathbf{p}^2\ \ \{u^2 = c^2, P^2 = m^2c^2\} \Rightarrow\ m^2c^2 = E^2/c^2 - \mathbf{p}^2\ \Rightarrow\ E^2 = m^2c^4 + \mathbf{p}^2c^2$$

- Special case of $m = 0$: $E = pc$ & $E = \gamma mc^2 \Rightarrow pc = \gamma mc^2 \Rightarrow p = \gamma mc = \gamma mv \Rightarrow v = c$

  => Massless particles must travel at the speed of light!

❖ NB: v here is not the speed v of S' w.r.t. the S ! Pay attention to the context

Relativistic kinematics: (Kinetic) energy, momentum



Speed v of S' w.r.t. the S along z-axes

$$\beta = v/c\ ,\quad \gamma = 1/\sqrt{1-\beta^2}$$

Lorentz boost. γ – Lorentz factor

- Similar to space-time, the 4-momenta in two frames are related through LT

$$\begin{aligned} p'_x &= p_x & p_x &= p'_x \\ p'_y &= p_y & p_y &= p'_y \\ p'_z &= \gamma\,(p_z - \beta E/c) & p_z &= \gamma\,(p'_z + \beta E/c) \\ E'/c &= \gamma\,(E/c - \beta p_z) & E/c &= \gamma\,(E'/c + \beta p_z) \end{aligned}$$

- Useful formulae $E = \gamma m c^2$, $\mathbf{p} = \gamma m \mathbf{v}$, $E^2 = m^2c^4 + \mathbf{p}^2c^2$, $K = E - mc^2 = (\gamma - 1)mc^2$
- In natural units $E = \gamma m$, $E^2 = m^2 + \mathbf{p}^2$, $K = (\gamma - 1)m$, $\gamma = E/m$, $\mathbf{p}^2 = K^2 + 2Km$

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