Relativity and GPS: How Einstein Helps You Find Your Way Home

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Who here has used a navigation program in your phone or your car within the past week?

- If the corrections specified by Einstein's Special and General Theories of Relativity were not applied to the calculations in your navigation programs, their error would be 7.3 miles/day = 11.7 km/day.
- To understand those corrections, let's look at
  (1) the principle of relativity,
  (2) the Lorentz transformations of special relativity, and
  (3) the general relativity gravitational time correction.

Einstein did not invent relativity!

Galileo Galilei (1564 – 1642)

Galilean Relativity

To show that Earth could be moving, Galileo proposed the following experiment:

Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it. With the ship standing still observe carefully how the little animals fly with equal speed to all sides of the cabin. The fish swim indifferently in all directions; the drops fall into the vessel beneath; and in throwing something to your friend, you need throw it no more strongly in one direction than in another, the distances being equal; jumping with your feet together you will pass equal spaces in every direction. When you have observed all these things carefully...have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change is all the effects named, nor could you tell from any of them whether the ship was moving or standing still.

(G. Galilei, *Dialogue Concerning the Two Chief World Systems*, S. Drake, trans., University of California Press, Berkley, 1962, pp. 186-187.)

Public Test in 1641

From the official record of the demonstration:

Mr. Gassendi, always having been curious to seek to justify by experiments the truth of the speculations proposed to him by philosophy and finding himself in Marseilles with his Lordship the Count of Allais in the year 1641, demonstrated, on a galley which set out to sea designedly by order of this Prince,... that a stone dropped from the very top of the mast, while the galley is sailing with all force and speed possible, will not fall in any other spot than it would if this same galley were stopped and immobile.

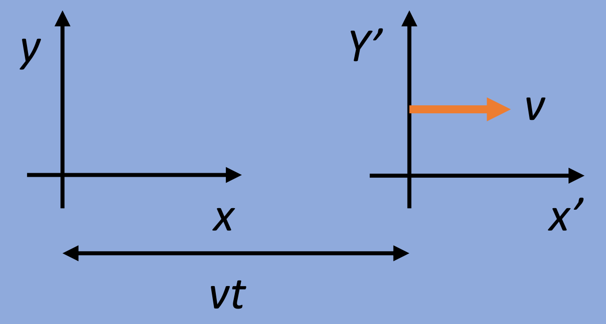
(R. Dugas, *Mechanics in the Seventeenth Century*, F. Jacquot, trans., Éditions du Griffon, Neuchatel, 1958, p. 110. Also quoted in A. Koyre, *Metaphysics and Measurement*, Harvard University Press., Cambridge, 1968, pp. 126-127.)

Galilean Relativity

Mechanical experiments, like Gassendi's falling stone, cannot distinguish between reference frames at rest and reference frames moving with constant velocity relative to a frame at rest.

Galilean Transformations

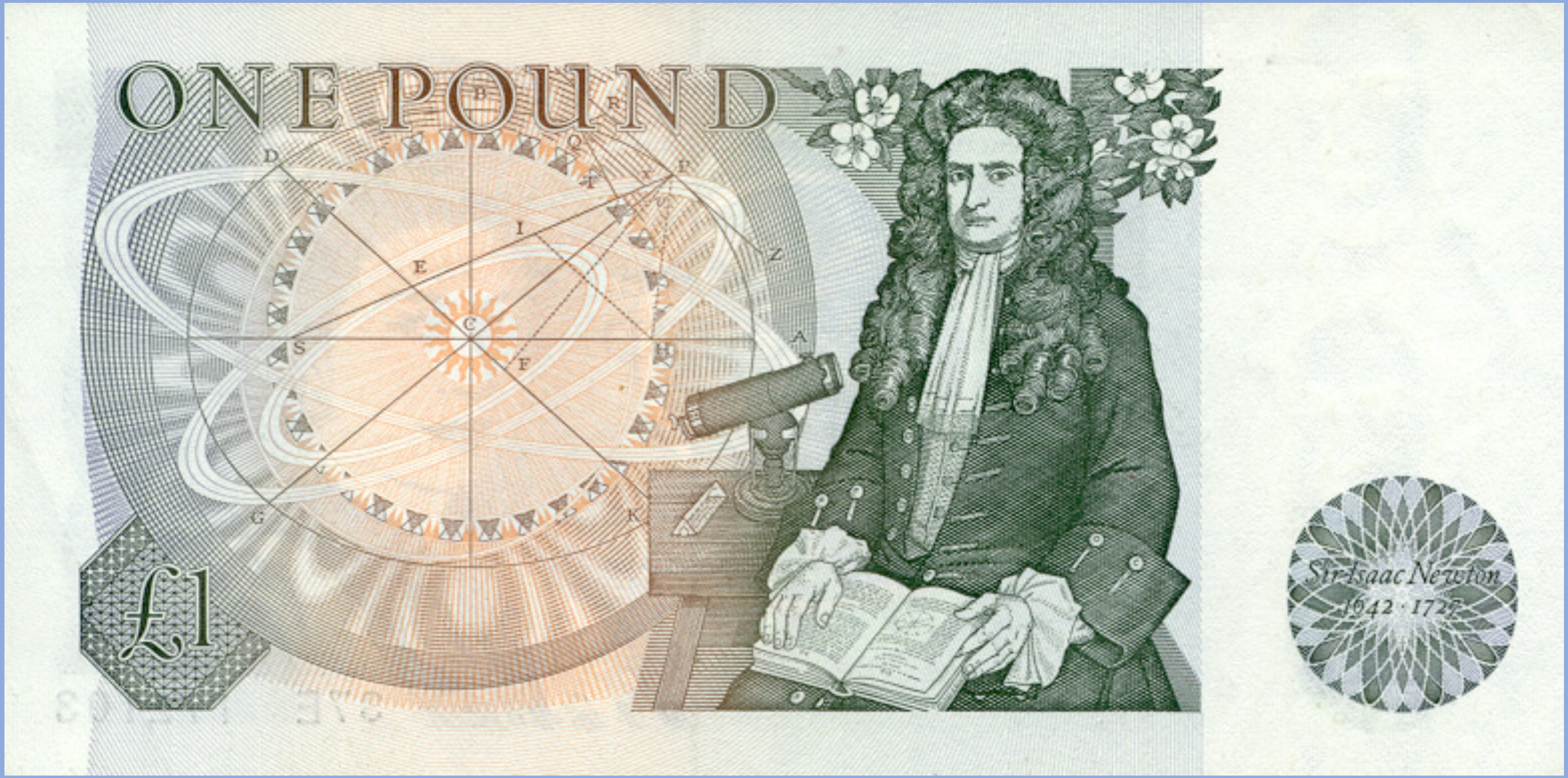
- Transformations relate measurements of an event taken in one coordinate system to the corresponding measurements of the event taken in another coordinate system, *e. g.* the position of an object measured from true north to the position measured from magnetic north.
- Galilean transformations relate measurements of event coordinates (time and position) in two inertial reference frames moving at constant velocity relative to each other. Origins and axes overlap at time $t = t' = 0$.
  The Rocket system (t', x', y', z') moves with constant velocity v in the +x-direction of the Lab system (t, x, y, z).



Galilean Transformations:

$t = t', \quad x = x' + vt', \quad y = y', \; z = z'$

Isaac Newton (1642 – 1727)



Newtonian Relativity In Mechanics

- "Absolute, true, and mathematical time," Newton said, "flows equably without relation to anything external," and "Absolute space … without relation to anything external remains always similar [homogeneous] and immovable." Newton's absolute time flows uniformly at all locations, and his absolute space remains forever stationary and uniform in all dimensions. However, as Newton acknowledged, absolute time and absolute space are neither observable nor measurable. Time is actually measured by counting repetitions of some observable motion, and space is measured relative to a reference frame marked by observable objects. Newton called these measurable quantities "relative, apparent, or common time" and "relative space."

(I. Newton, *Mathematical Principles of Natural Philosophy*, A. Motte, trans., revised by F. Cajori, University of California Press, Berkeley, 1962, p. 6.)

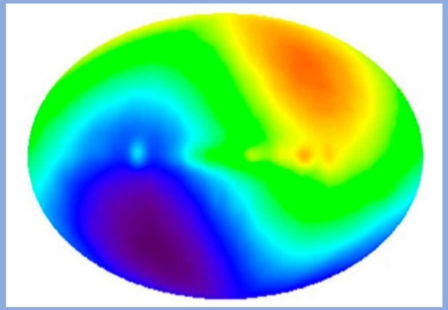
- Reference frames in which Newton's inertia law (Law I) holds are called inertial reference frames. Any reference frame moving with constant vector velocity relative to an inertial reference frame is also an inertial frame. Or, as Newton stated in his generalization of Galileo's relativity principle in Corollary V of his Laws of Motion:

"The motions of bodies included in a given space are the same among themselves, whether that space is at rest, or moves forwards in a right line without any circular motion." (I. Newton, *ibid*. p.20.)

Newton extended this idea in Corollary VI to include frames in which all bodies are moved with equal acceleration in straight lines, as in gravitational free fall or in the International Space Station.

Reference Frames

- On 1 January 1998 the International astronomical Union established the, presumably, non-rotating International Celestial Reference System, realized by three successive International Celestial Reference Frames in which the barycenter of the Solar System is fixed with reference to many distant radio sources (mostly quasars) measured with milliarcsecond precision and corresponding optical sources measured by the Gaia satellite. The most recent frame (1 January 2022) is specified by ICRF3 and Gaia-CRF3.
- A Planck satellite research group has measured the 370 km/s speed of the Sun relative to the Cosmic Microwave Background (CMB) toward the constellation Crater near the border of Leo and Virgo. They deduced this speed by examining the dipole term in the CMB radiation distribution over space.



How inertial are we?

- Earth rotation: $r_E = 6.38 \times 10^6$ m; $T_E = 86164$ s (sidereal day)
  $v_0 = 465$ m/s at equator, $v_{42} = 346$ m/s at 42 deg N
  $a_0 = ?$ $a_{42} = ?$
- Earth orbit around Sun: $r_{Eorb} = 1.5 \times 10^{11}$ m = 1 AU $T_{Eorb} = 3.16 \times 10^7$ s = 1 yr
  $v_{Eorb} = 29.8$ km/s $= 3 \times 10^4$ m/s
  $a_{Eorb} = ?$
- Sun orbit around Galaxy: $r_{Sorb} = 2.53 \times 10^{20}$ m $= 2.67 \times 10^4$ ly
  $T_{Sorb} = 7.14 \times 10^{15}$ s $= 2.26 \times 10^8$ yr
  $v_{Sorb} = 223$ km/s $= 2.23 \times 10^5$ m/s
  $a_{Sorb} = ?$

The solar motion around the Galaxy is nearly opposite in direction to 370 km/s solar motion through the CMB.

How inertial are we?

- Earth rotation: $r_E = 6.38 \times 10^6$ m, $T_E = 86164$ s (sidereal day)
  $v_0 = 465$ m/s at equator, $v_{42} = 346$ m/s at 42 deg N
  $a_0 = 0.034$ m/s^2, $a_{42} = 0.025$ m/s^2
- Earth orbit around Sun: $r_{Eorb} = 1.5 \times 10^{11}$ m = 1 AU, $T_{Eorb} = 3.16 \times 10^7$ s = 1 yr
  $v_{Eorb} = 29.8$ km/s = 3×10^4 m/s
  $a_{Eorb} = 0.006$ m/s^2 = 6×10^{-3} m/s^2
- Sun orbit around Galaxy: $r_{Sorb} = 2.53 \times 10^{20}$ m = 2.67×10^4 ly
  $T_{Sorb} = 7.14 \times 10^{15}$ s = 2.26×10^8 yr
  $v_{Sorb} = 223$ km/s = 2.23×10^5 m/s
  $a_{Sorb} = 2.0 \times 10^{-10}$ m/s^2

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A 19th Century Question from Optics

Could the luminiferous (*i. e.* light-bearing) aether serve as a stationary reference frame in absolute space?

(Note: "Aether"is often spelled "ether." I use "aether" to correspond to the original Greek spelling and to avoid confusion with chemical ethers, *e. g.* $CH_3CH_2OCH_2CH_3$.)

In Greek mythology, Aether was the son of Erebus (darkness) and Nyx (night). He was the personification of the pure essence breathed by the gods.

For Aristotle, aether was the fifth element (translated as "quintessence" in Latin) that constituted the heavens. According to Aristotle, the natural circular motion of the aether carried the heavenly bodies in their motions around Earth.

Luminiferous Aether as a Mechanical Medium

Huygens’ Aether

In his *Treatise on Light* (1690), Christiaan Huygens emphasized the analogy between light and sound:

“[W]hen one considers the extreme speed with which light spreads on every side, and how, when it comes from different regions, even those directly opposite, the rays traverse one another without hindrance, one may well understand that when we see a luminous object, it cannot be my any transport of matter coming to us from this object. . . . It will follow that this movement, impressed on the intervening matter, is successive; and consequently it spreads, as Sound does, by spherical surfaces and waves”

(C. Huygens, *Treatise on Light*, S. Thompson, trans., Dover Publications, New York, 1962, pp. 3-4.)

“[T]his matter. . .in which the movement coming from the luminous body is propagated, which I call Ethereal matter, . . . is not the same that serves for the propagation of Sound.” (*Ibid*., p. 11.)

Huygens conceived light vibrations as longitudinal vibrations due to elastic particle collisions in analogy to sound vibrations in air, but he noted that the ethereal matter must exist in vacuum spaces at the tops of barometers and in the region between Earth, Sun, and stars.

Newton's Aether

- Huygens' wave theory of light propagation was neglected in the 1700s in favor of Newton's suggestion in Query 29 of his *Opticks*: "Are not the Rays of Light very small Bodies emitted from shining Substances?" (I. Newton, *Opticks*, 4th ed., 1730, Dover Publications, New York, 1952, p. 370.)
- Newton, however, did imagine an aethereal medium that influences light particles and transmits heat vibrations through vacuum (Query 18), produces refraction by changes in density (Query 19), produces diffraction (Query 20), causes gravitational attraction (Query 21), but does not inhibit celestial motions (Query 22) ["If this *Aether* (for so I call it) should be supposed 700000 times more elastic than our Air, and above 700000 times more rare; its resistance would be 600000000 times less that that of Water. And so small a resistance would scarce make any sensible alteration in the Motions of the Planets in ten thousand years." (*Ibid*. pp. 352-353.)], and excites vision (Query 23).

Luminiferous Aether

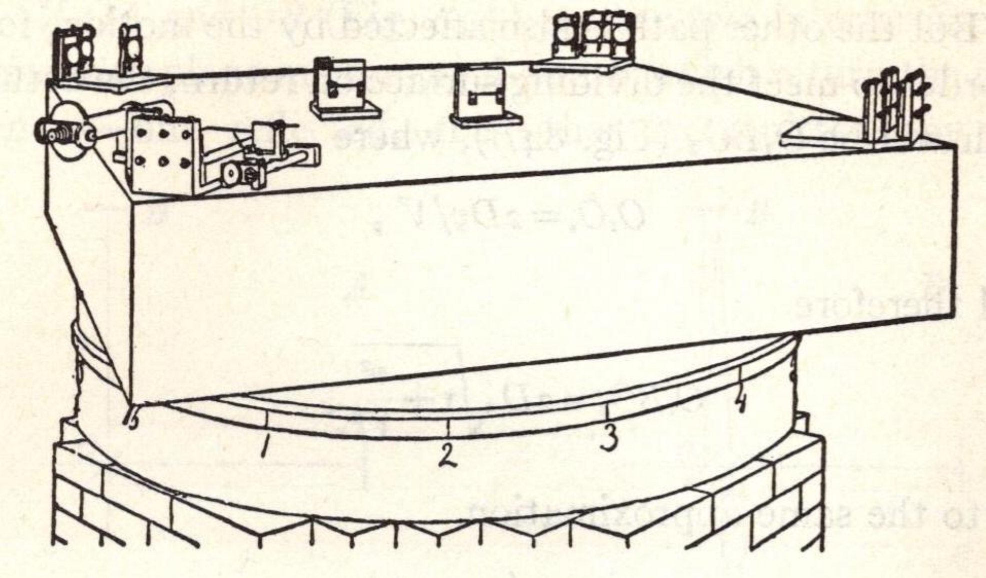
- With the revival of the wave model of light by Thomas Young (1803) to explain double-source interference phenomena and by Augustin-Jean Fresnel (1815-1818) to explain diffraction, the idea of an aether medium to carry light vibrations was reconsidered.
- James Clerk Maxwell (1864) developed a theory of electromagnetic (EM) vibrations in the aether in which the speed EM waves equaled the speed of light: $c = \sqrt{{}^{2\mathrm{k}_C}/_{\mathrm{k}_A}} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \cong 3\mathrm{x}10^8\ \mathrm{m/s}.$ Maxwell identified the EM fields with aether at rest in any inertial frame.
- Heinrich Hertz (1887-1888) demonstrated the existence of Maxwell’s electromagnetic waves and confirmed they traveled at speed of light.

Problems with a Mechanical Aether

- To transmit light through all space, the aether must fill astronomical space, which suggested that it was an expansive fluid.
  But aether must be solid to support transverse waves, as indicated by polarization phenomena.
- To support the rapid speed of light, a mechanical aether must be highly rigid.
  But aether must have nearly zero resistance and viscosity relative to the astronomical motions of moons, planets, stars.

Problems with an Optical Aether

- James Bradley's observations of stellar aberration (1729) suggested that Earth moved through an aether stationary relative to the Sun. (Note umbrella and raindrop analogy.)
- Michelson and Morley's experiment (1887) suggested that aether was stationary relative to their apparatus (no significant fringe shift as apparatus rotated).



Initial Efforts at Reconciliation

- George FitzGerald (1889) suggested that all objects shrink in the direction of motion through the aether by the ratio $\sqrt{1-\frac{v^2}{c^2}}$, where v = speed of Earth in orbit and c = speed of light.
- Hendrik Lorentz (1902-1904) developed relations, based on his theory of electrons, to show "that many electromagnetic actions are entirely independent of the motion of the system [for $v<c$]." (*The Principle of Relativity*, 1923, Dover Publications, New York, p. 13.)
  Those relations between space and time measurements in different inertial frames are now called Lorentz transformations. They account for FitzGerald's length contraction and add time dilation, as well.

Albert Einstein’s Special Relativity

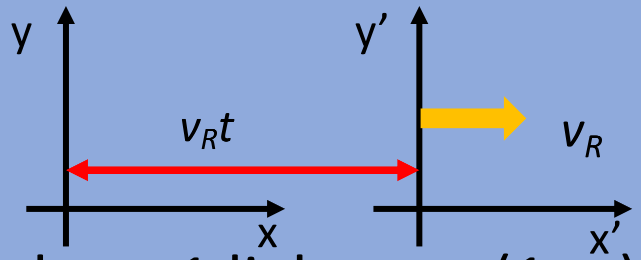
- In 1905, in the first part of his paper “On the Electrodynamics of Moving Bodies” (*The Principle of Relativity*, 1923, Dover Publications, New York, pp. 37-48.), Einstein independently derived the Lorentz transformations From two simple assumptions:
  (1) “the Principle of Relativity,” *i. e.* All physics laws remain the same in each inertial reference frame. and
  (2) “light is always propagated in empty space with a definite velocity *c* which is independent of the state of motion of the emitting body.” (Ibid., p. 38), *i. e.* Maxwell’s equations are among accepted laws of physics.
- Consequently, “[t]he introduction of a ‘luminiferous ether’ will prove superfluous.” (Ibid.)

Lorentz Transformations

- Many AP physics texts give formulas for applying results of Lorentz transformations, *e. g.* length contraction and time dilation, but they don’t give the reasoning that leads to the derivations.
- There are many different derivations. The worksheet I sent to you, based on the derivation found in *Spacetime Physics* by Edwin Taylor and John Wheeler, is, I think, particularly clear.
- BREAK TO REVIEW LORENTZ TRANSFORMATION DERIVATION

Lorentz Transformations

Inertial Reference Frames: origins and axes overlap at $t = t' = 0$.



Lab frame is stationary. Rocket frame moves with speed v_R in +x-direction.
Units are chosen so that $c = 1$, *e. g.* distance in m and time in light-meters,
where 1 light-m = (1 m)/c = 3.3 ns, $v_R<1$ is measured as a fraction of c.

Let $\gamma = (1 - v_R^2)^{-1/2}$ Note $\gamma \geq 1$.

$t = \gamma(v_R x' + t')$ $\quad x = \gamma(x' + v_R t')$ $\quad y = y'$ $\quad z = z'$ $\qquad t^2 - x^2 = t'^2 - x'^2 = \tau^2$

$t' = \gamma(-v_R x + t)$ $\quad x' = \gamma(x - v_R t)$ $\quad y' = y$ $\quad z' = z$ $\quad \tau$ = *proper time between events*

Proper time between events is invariant, *i. e.* the same in all inertial frames.
If a clock at the Lab frame origin ($x = 0$) has time between tics = $t = \tau$. As viewed in the Lab frame, an identical clock at the Rocket frame origin has time between tics $t' = \gamma t$, *i. e.* the Rocket clock appears to tic more slowly than the Lab clock.

Galilean Transformations

Note that when v_R<<1 (c = 1), $\gamma \cong 1$ and the Lorentz transformations are approximated by $t = 1(v_R x' + t')$ and $x = 1(x' + v_R t')$
or $t = t'$ and $x = x' + v_R t'$.

Recall for most measurements t' (light-m) is vastly greater than x' (m), and since v_R<<1, we have $v_R x' <<< t'$. Thus, event times are approximately the same in both frames.

Notice that the Galilean transformations approximate the Lorentz transformations for low velocities of relative motion compared to c.

Time Dilation

The 1962 film "Time Dilation: An Experiment with Mu-Mesons" (35:40)
(https:www.youtube.com/watch?v=rbzt8gDSYIM)
was made when muons were still called mu-mesons.

The experiment with cosmic ray muons presents a dramatic demonstration of extreme time dilation, as viewed in the Earth frame, and extreme length contraction, as viewed in the muon frame.

Gravitational Influence on Clocks

Einstein proposed his General Relativity (GR) in 1915 to account for accelerating reference frames. Karl Schwarzschild in 1916 found a solution to Einstein's field equations for the case of a spherically symmetric, non-spinning, non-electrically charged mass. The solution relates measurements of space and time coordinates of events that occur in space at a distance *r* from the center of the mass *M*. The time between tics (t_r) of a clock at *r* compared to the time between tics (t_∞) of a clock a great distance ($r = \infty$) from the mass is given by $\frac{t_r}{t_\infty} = \sqrt{\left(1 - \frac{2GM}{c^2 r}\right)}$, where *G*, *M*, *r*, *c*, and both *t* values are in SI units.

Black Holes

Note that the expression $\frac{t_r}{t_\infty} = \sqrt{\left(1 - \frac{2GM}{c^2 r}\right)}$ goes to zero when $r_{EH} = \frac{2GM}{c^2}$. The r_{EH} value is the radius of the Event Horizon around a Schwarzschild black hole of mass *M*. Calculate r_{EH} for the following:

Earth Mass $M_{\mathrm{E}} = 6 \times 10^{24}$ kg $\quad r_{\mathrm{EEH}} = ?$

Sun Mass $M_{\mathrm{S}} = 2 \times 10^{30}$ kg $= 3.3 \times 10^{5}\ M_{\mathrm{E}}$ $\quad r_{\mathrm{SEH}} = ?$

SgrA* Mass $M_{\mathrm{SgrA*}} = 4 \times 10^{6}\ M_{\mathrm{S}}$ $\quad r_{\mathrm{SgrA*EH}} = ?$

Black Holes

Note that the expression $\frac{t_r}{t_\infty} = \sqrt{\left(1 - \frac{2GM}{c^2 r}\right)}$ goes to zero when $r_{EH} = \frac{2GM}{c^2}$. The r_{EH} value is the radius of the Event Horizon around a Schwarzschild black hole of mass *M*. Calculate r_{EH} for the following:

Earth Mass $M_E = 6 \times 10^{24}$ kg $r_{EEH} = 9 \times 10^{-3}$ m = 1 cm

Sun Mass $M_S = 2 \times 10^{30}$ kg = $3.3 \times 10^5\ M_E$ $r_{SEH} = 2.9 \times 10^3$ m = 2.9 km

SgrA* Mass $M_{SgrA*} = 4 \times 10^6\ M_S$ $r_{SgrA*EH} = 1.2 \times 10^{10}$ m = 0.1 AU

Navigation Programs

- Have you noticed that your phone gets warm when using a navigation program? Why is that?
- How does the navigation program know where you are?
- What is GPS?
- The 2019 video "How Does GPS Actually work and Why Many GPS Devices are About to Stop Working" (14:35) gives a brief history of the navigation system and how it works. It omits reference to relativistic corrections. (https://www.youtube.com/watch?v=CnwAJrDikgU)

Google Maps and GPS

- Google Maps combines a set of maps, images, and other information about places on Earth with current location information determined by your phone acting as a GPS receiver.
- The Global Positioning System (GPS) is a constellation of at least 24 satellites (currently 31, including spares). The satellites, maintained by the U. S. Space Force, have been placed in orbits at an altitude of about 20,200 km above Earth or 26,580 km from Earth's center. The orbital period is 12 hours, which you can calculate from Kepler's law of periods or Newton's gravitation law. Satellite orbits are inclined at 55 degrees to Earth's equator so that at least four are above the horizon at any one time at almost any place on Earth. The first group of satellites, which are no longer operational, was launched in the 1990s. New groups are regularly launched with updated features. The U. S. Space Force continually tracks the satellites with ground-based radars to verify their positions, synchronize their clocks, and update their onboard ephemeris equations and almanac data that are used to broadcast the satellite positions

GPS and Relativity

The following analysis is adapted from the Wikipedia article "Error analysis for the Global Positioning System"
https://en.wikipedia.org/wiki/Error_analysis_for_the_Global_Positioning_System#:~:text=In%20the%20context%20of%20GPS,body)%20appear%20to%20tick%20slower.

Kinetic time dilation

The factor by which clocks in the GPS satellites tick slower, due to satellite velocity, than clocks stationary on Earth is determined using the Lorentz transformation. Time measured by an object with velocity *v* compared to a stationary object is given by (the inverse of) the Lorentz factor, γ:

$$\frac{t_{GPS}}{t_{Earth}} = \frac{1}{\gamma} = \sqrt{1 - v^2/c^2}$$

For small values of *v/c,* this ratio, by the binomial approximation, is approximately:

$$\frac{t_{GPS}}{t_{Earth}} = \frac{1}{\gamma} \cong 1 - \frac{v^2}{2c^2}$$

The GPS satellites, with orbit radius about 26580 km and 12-hour orbit period, move at about 3874 m/s relative to Earth's center.

Kinetic Time Dilation (2)

We thus calculate:

$$\frac{t_{GPS}}{t_{Earth}} = \frac{1}{\gamma} \cong 1 - \frac{v^2}{2c^2} \cong 1 - \frac{(3874\ \mathrm{m/s})^2}{2(2.998 \times 10^8\ \mathrm{m/s})^2} \cong 1 - 8.349 \times 10^{-11}$$

This value of -8.349×10^{-11} represents the difference in the rate by which the GPS satellite clocks tick slower than Earth-stationary clocks. That rate difference multiplied by the number of nanoseconds in a day yields the nanoseconds per day lost by GPS clocks relative to Earth clocks due to satellite speed:

$(-8.349 \times 10^{-11})(86400\ \mathrm{s/day})(10^9\ \mathrm{ns/s}) \cong -7214\ \mathrm{ns/day}$

In other words, the GPS satellite clocks are slower than clocks at rest on Earth by 7214 nanoseconds per day due to GPS satellite clock velocity.

Kinetic Time Dilation (3)

Note that this speed of 3874 m/s as measured relative to Earth's center rather than its surface where the GPS receivers (and users) are. This is because Earth's equipotential makes net time dilation equal across its geodesic surface. That is, the combination of Special and General effects makes the net time dilation at the equator equal to that of the poles, which in turn are at rest relative to the center. Hence, we use the center as a reference point to represent the entire surface.

Gravitational Time Dilation

Gravitational Time Dilation

The Schwarzschild metric gives the relation between time kept by a stationary clock at distance *r* from a spherical mass *M* compared to the time kept by a stationary clock far away $(r \cong \infty)$ from the mass.

$$\frac{t_r}{t_\infty} = \sqrt{\left(1 - \frac{2GM}{c^2 r}\right)}$$

where t_r is the time passed between events, e. g. clock tics, measured by a clock at a distance r from the center of the Earth and t_∞ is the time passed between the events as measured by a far-away observer. *G* is the Newtonian gravitation constant, and *M* is the Earth mass for the case of Earth and GPS satellites.

For small values of $GM/(c^2 r)$ this ratio is approximately:

$$\frac{t_r}{t_\infty} \cong 1 - \frac{GM}{c^2 r}$$

Gravitational Time Dilation (2)

The clocks in the GPS satellites orbiting in a weaker gravitational field at a distance of about 4.2 Earth radii from Earth's center tic faster than identical clocks on Earth by a ratio: $t_{\text{GPS}}/t_{\text{Earth}}$:

$$\frac{t_{GPS}}{t_{Earth}} = \frac{t_{GPS}/t_{\infty}}{t_{Earth}/t_{\infty}} \cong \left(1 - \frac{GM}{c^2 r_{GPS}}\right)\left(1 - \frac{GM}{c^2 r_{Earth}}\right)^{-1} \cong \left(1 - \frac{GM}{c^2 r_{GPS}}\right)\left(1 + \frac{GM}{c^2 r_{Earth}}\right)$$

$$\frac{t_{GPS}}{t_{Earth}} \cong 1 + \left(\frac{GM}{c^2 r_{Earth}} - \frac{GM}{c^2 r_{GPS}}\right) \cong 1 + 5.307 \text{ x } 10^{-10},$$

for r_{Earth} = 6,357,000 m, r_{GPS} = 26,541,000 m, Earth $M = 5.974\times10^{24}$ kg,

$G = 6.674\times10^{-11}$ m^3 kg^{-1}s^{-2}, and $c = 2.998\times10^{8}$ m/s.

Gravitational Time Dilation (3)

The value 5.307 x 10^{-10} represents the fraction by which the clocks at GPS satellite’s altitude tick faster than identical clocks on the surface of Earth. This fraction multiplied by the number of nanoseconds in a day yields the nanoseconds per day gained by GPS clocks relative to Earth clocks due to the difference in the local gravitational field:

$$(+5.307 \times 10^{-10})(86400 \text{ s/day})(10^{9} \text{ ns/s}) \cong +45850 \text{ ns/day}$$

Thus, the satellites' clocks gain 45850 nanoseconds a day due to gravitational time dilation.

Combined Time Dilation Effects

These effects are added together to give (rounded to 10 ns):

45850 – 7210 = 38640 ns/day

Hence, the satellites' clocks gain approximately 38,640 nanoseconds a day or 38.6 μs per day due to relativistic effects in total.

To compensate for this gain, a GPS clock's frequency needs to be slowed by the fraction:

$5.307\times10^{-10} - 8.349\times10^{-11} = 4.472\times10^{-10}$

This fraction is subtracted from 1 and multiplied by the pre-adjusted clock frequency of 10.23 MHz:

$(1 - 4.472\times10^{-10}) \times 10.23 = 10.22999999543$

In other words, we need to slow the clocks down from 10.23 MHz to 10.22999999543 MHz to negate both time dilation effects.

Sources of User Equivalent Range Errors

Source	Effect (m)
Signal arrival C/A	±3
Signal arrival P(Y)	±0.3
Ionospheric effects	±5
Ephemeris errors	±2.5
Satellite clock errors	±2
Multipath distortion	±1
Tropospheric effects	±0.5
$3\sigma R$ C/A (code)	±6.7
$3\sigma R$ P(Y) (code)	±6.0

Conclusion

Computer navigation systems using GPS or other satellite navigation systems confirm the predictions of Einstein's special and general relativity theories every time you use them to find your location or to find your way home.

Thank you, Albert!