QM, The Uncertainty Principle + Virtual Particles

The story begins in 1900 with 'Max Planck' describing the Black Body Spectrum: he assumes that the cavity is composed many little oscillators

$$E = h\nu = \frac{h}{2\pi}\omega = \hbar\omega$$

- not like macroscopic oscillators E is not related to $\nu(\omega)$
- is correct for quantum oscillators

In 1905, Einstein realized that if the EM field was composed of corpuscles in thermal equilibrium with the cavity walls, then the corpuscles had energies

$$E = \hbar\omega$$

He then used this to explain the Photoelectric Effect

In 1925, Louis de Broglie hypothesized that all matter obeyed wave properties. Using the Einstein-Planck relation and Special Relativity, he showed that

$$\lambda = \frac{h}{p} \Rightarrow p = \frac{2\pi\hbar}{\lambda} = \hbar k$$

where k is the spatial wavenumber

Two years later, this hypothesis was verified by Thomson (x-rays) + Davisson-Germer (electrons)

A traveling matter wave has the form (in 1-D)

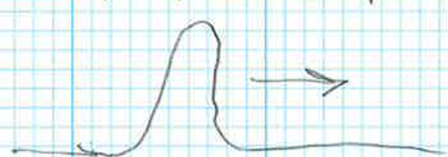
$$\psi = A e^{i(kz-\omega t)} = A e^{\frac{i}{\hbar}(pz - Et)}$$

The probability of finding a particle somewhere is

$$\text{Prob} \sim |\psi|^2$$

But wait: can particles be described by a wave that fills all of space

Particles are localized phenomena



$\Rightarrow$ Add many waves of different frequencies + wavelengths

$\Rightarrow$ Fourier transform

$$\psi(\ ) = \int_{-\infty}^{\infty} dk' \, \bar{\Psi}(k') \, e^{ik'z} \cdot \int_{-\infty}^{\infty} \bar{\Psi}(\omega') \, e^{-i\omega' t}$$

need a range of k

where $$\bar{\Psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz' \, e^{-ikz'} \, \psi(z')$$

Let's take a Gaussian Wave Packet

$$\psi(z) = \frac{1}{\sigma_z \sqrt{2}} e^{-z^2/2\sigma_z^2}$$

$$\psi(t) = \frac{1}{\sigma_t \sqrt{2}} e^{-t^2/2\sigma_t^2}$$

Doing the integrals, we get

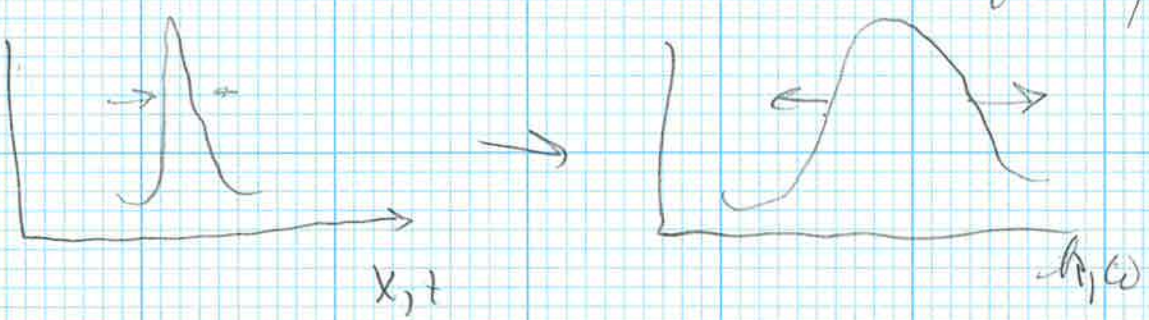
$$\tilde{\psi}(k) = \frac{\sqrt{2}\,\sigma_z}{\sqrt{2\pi}}\, e^{-(\sigma_z^2 k^2/2)} = \frac{1}{\sigma_k\sqrt{2\pi}}\, e^{-(k^2/2\sigma_k^2)}$$

$$\tilde{\psi}(\omega) = \frac{\sqrt{2}\,\sigma_t}{\sqrt{2\pi}}\, e^{-(\sigma_t^2 \omega^2/2)} = \frac{1}{\sigma_\omega\sqrt{\pi}}\, e^{-\left(\frac{\omega^2}{2\sigma_\omega^2}\right)}$$

We see that the packet widths are inversely related

$$\sigma_k = \frac{1}{\sigma_z}, \quad \sigma_\omega = \frac{1}{\sigma_t}$$

If our particle is well in space + time, it is broadly distributed in wave number + frequency



This is a property of ALL waves

$$\sigma_k \sigma_z = 1 \qquad \sigma_\omega \sigma_t = 1$$

$$\underbrace{\Delta k}_{\frac{\Delta P}{\hbar}} \Delta z = 1 \qquad \underbrace{\Delta \omega}_{\frac{\Delta E}{\hbar}} \Delta t = 1$$

$$\Delta P \Delta z = \hbar \qquad \Delta E \Delta t = \hbar$$

Uncertainty Principle

consequence of waves

A traveling wave has the form

$$\psi = A e^{i(\vec{k}\cdot\vec{x}-\omega t)} = A e^{\frac{i}{\hbar}(\vec{p}\cdot\vec{x}-Et)}$$

The probabity of finding a particle goes like $|\psi|^2$

In QM, the waves (wavefunctions) can be operated on by Operators

$$\hat{P}\psi = p\psi \qquad \hat{E}\psi = E\psi$$

It's clear that

$$\hat{P} = \frac{\hbar}{i}\frac{\partial}{\partial x} = \frac{\hbar}{i}\nabla$$

$$\hat{E} = i\hbar\frac{\partial}{\partial t}$$

Schrodinger decided that a good wavefcn should satify the energy conservation condition,

$$H\psi = \left(\frac{\hat{P}^2}{2m} + V(x)\right)\psi = E\psi = i\hbar\frac{\partial\psi}{\partial t}$$

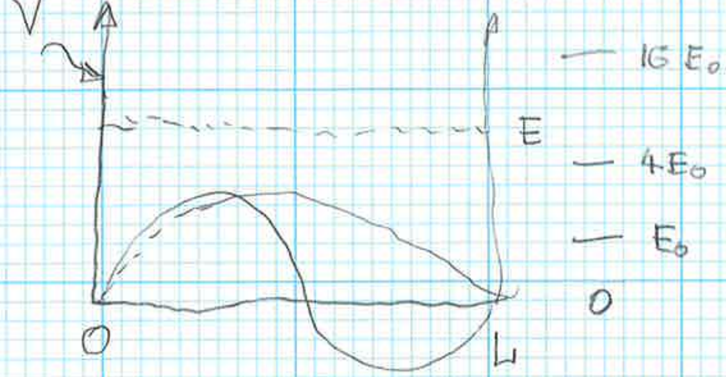
$$\Rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi = i\hbar\frac{\partial\psi}{\partial t}$$

Substituting our plane wave into the eq:

$$\frac{\hbar^2 k^2}{2m}\psi + V\psi = E\psi \Rightarrow \frac{k^2}{2m} + V = E$$

$$\Rightarrow k = \frac{\pm 1}{\hbar}\sqrt{2m(E-V)}$$

In the famous case of a particle in an infinite well,

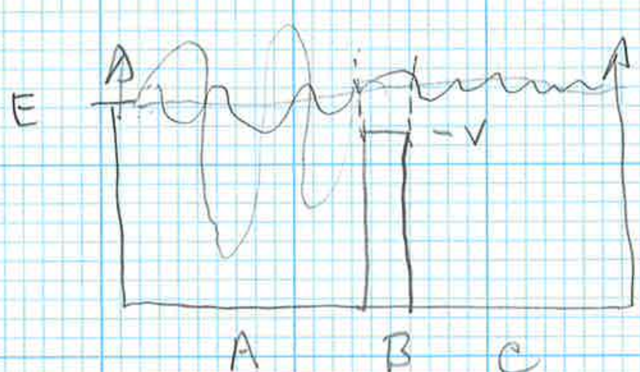


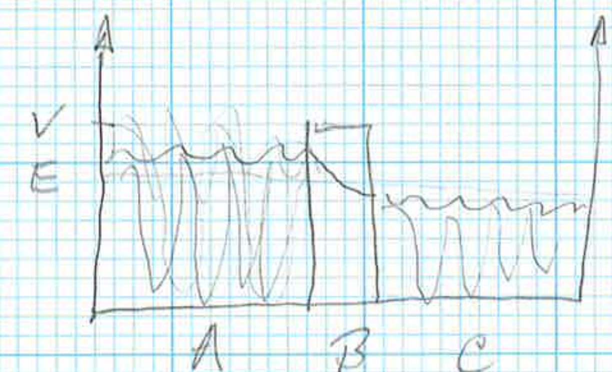
$$\psi = A\sin kx \qquad kL = n\pi \quad n = 1, 2, \ldots$$

the finite boundaries lead to discreet states

$$E = \frac{p^2}{2m} = \frac{\hbar^2\left(\frac{n\pi}{L}\right)^2}{2m} = \frac{n^2\pi^2\hbar^2}{2mL^2} = n^2 E_0$$

A more interesting case happens when there is a barrier of finite height in the middle





$k_{A,C} = \frac{1}{\hbar}\sqrt{2mE}$

$k_B = \frac{1}{\hbar}\sqrt{2m(E-V)}$

$k_{A,C} = \frac{1}{\hbar}\sqrt{2mE}$

$k_B = \frac{1}{\hbar}\sqrt{2m(E-V)} = \frac{i}{\hbar}\sqrt{2m(V-E)} = i\kappa$

$\psi = A e^{-\kappa x}$ ← exponential

- The particle has an exponentially suppressed but finite probability of penetrating the barrier!
- It is a Virtual Particle in Region B
  * Virtual means that it can't propagate "long distances"
  * Many examples of virtual whenever $E^2 \neq p^2 + m^2$
- Barrier penetration is an important phenomenon
  * transistors (exponential factor gives large gains)
  * fusion in stars
  * radioactive decay

What physically happens? What are momentum/energy in Region B?

In QM, conjugate variables like x, p that form "Uncertainty Principle", non-commuting operators,

$$\hat{x}\hat{p}\psi = \frac{\hbar}{i} x \frac{\partial \psi}{\partial x} \quad , \quad \hat{p}\hat{x}\psi = \frac{\hbar}{i}\left[\psi + \frac{\partial \psi}{\partial x}\right]$$

we can write that $[\hat{x}\hat{p} - \hat{p}\hat{x}]\psi = [\hat{x}, \hat{p}]\psi = i\hbar\psi$

$$[\hat{x}, \hat{p}] = i\hbar$$

In QM, a state cannot have definite values of both quantities unless their operators commute

The quantum harmonic oscillator

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2} k \hat{x}^2 = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega_0^2 \hat{x}^2$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

Solving Schrodinger's Eq $H\psi = E\psi$,

$$\psi_n = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right) \quad n = 0, 1, 2, \ldots$$

H_n = Hermite Polynomial

$$E_n = \hbar\omega_0 \left(n + \tfrac{1}{2}\right) \quad n = 0, \ldots,$$

$E_0 > 0$

equally spaced

Paul Dirac invented a really cool way to solve this problem: Define operators a and a^+,

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i}{m\omega} \hat{p}\right)$$

$$\hat{a}^+ = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i}{m\omega} \hat{p}\right)$$

$$[a, a^+] = \frac{m\omega}{2\hbar} \left(\frac{i}{m\omega} \underbrace{[\hat{p}, \hat{x}]}_{-i\hbar} - \frac{i}{m\omega} \underbrace{[\hat{x}, \hat{p}]}_{i\hbar} \right) = 1$$

You can show

$$a\psi_0 = 0 \quad \Rightarrow \quad a|0\rangle = 0$$

$$a^+\psi_0 = \psi_1 \qquad a^+|0\rangle = |1\rangle$$

In general

$$a|n\rangle = \sqrt{n}\, |n-1\rangle$$

$$a^+|n\rangle = \sqrt{n+1}\, |n+1\rangle$$

Note that $a^+ a|n\rangle = a^+ \sqrt{n}\, |n-1\rangle = \sqrt{n}\sqrt{n}\, |n\rangle = n|n\rangle$

The operator $a^\dagger a$ counts the state number!

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega_0^2 \hat{x}^2 = (a^\dagger a + \tfrac{1}{2})$$

This counting algebra has had huge impact in Quantum Field theory to count particles.

In ordinary QM, a two particle state looks something like this,

$$\psi_{12}(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_1(x_1)\psi_2(x_2) + \psi_1(x_2)\psi_2(x_1)\right]$$

where the particles have a definite symmetry under interchange

Using Dirac's algebra, this can be simplified

$$|2\rangle = \psi_{12}(x_1, x_2)$$

counts number of particles

$$a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$[a, a^\dagger] = 1$$

If no particles $|0\rangle$ (vacuum)

$a^\dagger|0\rangle = |1\rangle \leftarrow$ $a^\dagger$ creates particles

$a\,|1\rangle = |0\rangle$ $\quad$ a destroys particles

This is called Second Quantization and allows the creation of field operators in QFT

Field Operators

— can be expressed as Fourier Transforms

scalar $\varphi(x,t) = \sum_k \left[a_k e^{i(kx-\omega t)} + a_k^\dagger e^{-i(kx-\omega t)} \right]$

vector $A^\mu(x,t) = \sum_k \left[\epsilon_k^\mu a_k e^{i(kx-\omega t)} + \epsilon^{\mu *} a_k^\dagger e^{-i(kx-\omega t)} \right]$

polarization vectors

$\Psi(x,t) = \sum_k \left[u(k) b_k e^{i(kx-\omega t)} + \bar{v}(k) d_k e^{i(kx-\omega t)} \right]$

Dirac Spinor

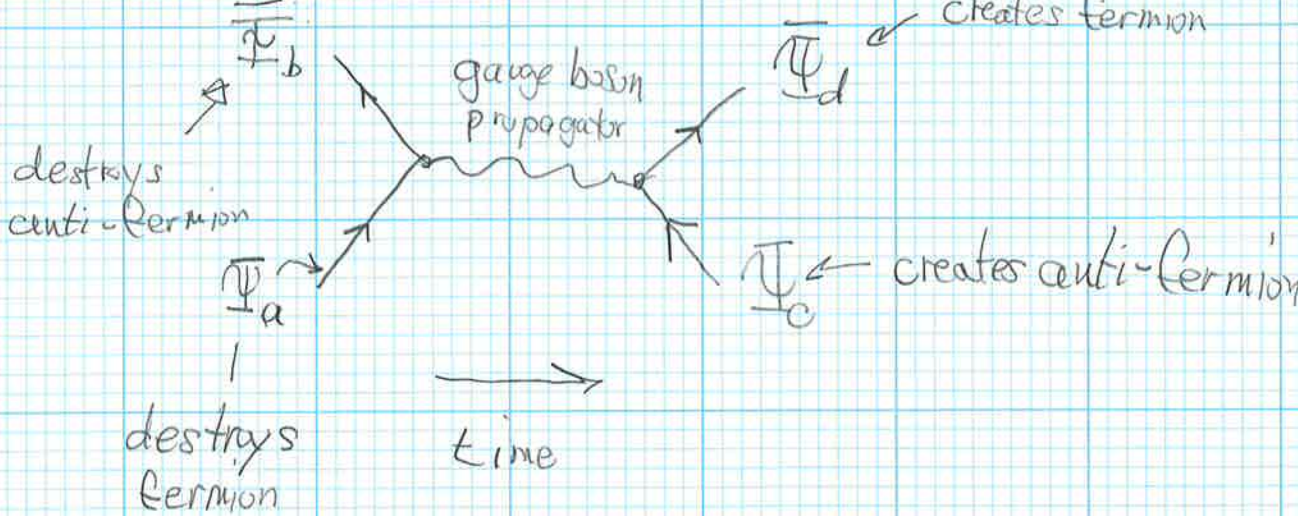
$[a_k, a_{k'}^\dagger] = \delta_{k,k'} \quad \{b_k, b_{k'}^\dagger\} = \delta_{k,k'}$

$a_k, a_k^\dagger$ ← destroy / create particles of momentum k

$b_k, b_k^\dagger$ are fermion operators, they anti-commute to get FD statistics [0, 1 particles / state]

Note that they create particles but also propagate as waves

- wave/particle duality is built-in
- not an either/or question
- these operators underly the Feynman diagrams that we draw all the time



Propagator $\quad D_{\mu\nu}(y-x) = \langle 0|T(A^*_\mu(y)\,A_\nu(x))|0\rangle$

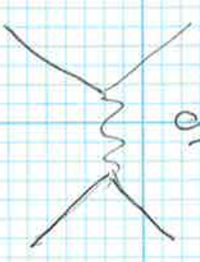
destroys γ at x and creates at y

In momentum (Fourier Transform) space, the propagator is proportional to

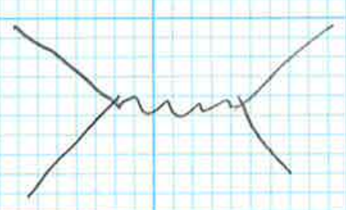
$$D(q^2) \sim \frac{1}{q^2 - m_V^2\,[+ i\sqrt{q^2}\,\Gamma_V]}$$

If $q^2 = E_V^2 - P_V^2 \neq m_V^2$ then the internal particle is virtual

This is almost alway true



$q^2 < 0$ always



except for produced W's + Z's this is true. Cannot produce massless photons

Examples of Virtual Photons,

static magnetic and electric fields

* a charged particle in a magnetic field curves
  - momentum changes
  - energy doesn't

$$q^2 = (\Delta E)^2 - (\Delta P)^2 = -(\Delta P)^2 < 0$$